Last update: 4:54pm, Jan 28.

Math 432: Set Theory and Topology HOMEWORK 1 **Due: Jan 28 (Tue)**

1. Let A,B,C be subsets of some ambient set U.

(a) Prove the *distributivity law*: $A\cap(B\cup C)=(A\cap B)\cup(A\cap C)$. Your proof may be a sequence of equivalences, but you have to justify each step.

(b) Deduce the identity $A\cup(B\cap C)=(A\cup B)\cap(A\cup C)$ from the distributivity law in the following two ways: using complements and not using complements (directly applying distributivity).

2. For sets A,B, we call the set $A\triangle B:=(A-B)\cup(B-A)$ their *symmetric difference*. For sets A,B,C, prove the following (using any method you like):

(a) $A\triangle B=(A\cup B)\setminus(A\cap B)$.

(b) $A\triangle C\subseteq(A\triangle B)\cup(B\triangle C)$.

3. Use induction to prove that a set with n elements has 2^n subsets.

4. For sets x,y, we define the *ordered pair* $(x,y):=\big\{\{x\},\{x,y\}\big\}$.

(a) Prove that this satisfies the main property of ordered pairs, namely: for any sets x_0,y_0,x_1,y_1, if $(x_0,y_0)=(x_1,y_1)$ then $x_0=x_1$ and $y_0=y_1$.

(b) Deduce that if $(x,y)=(y,x)$ then $x=y$.

5. Let X be a set and let $A_0,A_1\subseteq X$. Define a binary relation E on X by setting

$$x\,E\,y :\Longleftrightarrow \forall i\in\{0,1\}\ (x\in A_i\Leftrightarrow y\in A_i).$$

for $x,y\in X$.

(a) Prove that E is an equivalence relation.

(b) List all E-classes.

6. Let $\mathcal{Q}$ be a partition of a set X. Define a binary relation $R_{\mathcal{Q}}$ on X by:

$$x\,R_{\mathcal{Q}}\,y :\Longleftrightarrow \exists P\in\mathcal{Q} \text{ such that } x,y\in P,$$

for $x,y\in X$.

(a) Prove that $R_{\mathcal{Q}}$ is an equivalence relation.

(b) Show that the $R_{\mathcal{Q}}$-classes are exactly the sets in $\mathcal{Q}$, more precisely, $X/R_{\mathcal{Q}}=\mathcal{Q}$.

7. Terminology. For a function $f:A\to B$ and $A_0\subseteq A$, let $f|_{A_0}$ denote its *restriction to* A_0, namely, the function $f|_{A_0}:A_0\to B$ defined by $f|_{A_0}(a):=f(a)$ for each $a\in A_0$. When we say "f on A_0 has some property", we mean that $f|_{A_0}$ has that property.

Let $f:A\to B$ and $g:B\to C$.

(a) Prove that $g\circ f$ is injective if and only if f is injective and g is injective on $f(A)$ (i.e. $g|_{f(A)}$ is injective).

(b) Give an example of f and g such that f is injective yet $g\circ f$ is not.

(c) Prove that $g \circ f$ is surjective if and only if g is surjective on $f(A)$ (i.e. $g(f(A)) = C$).

(d) Give an example of f and g such that g is surjective yet $g \circ f$ is not.

8. Let $f : A \to B$ and $A_0, A_1 \subseteq A, B_0, B_1 \subseteq B$.

(a) Prove that f^{-1} respects unions, i.e. $f^{-1}(B_0 \cup B_1) = f^{-1}(B_0) \cup f^{-1}(B_1)$.

(b) Prove that f^{-1} respects complements, i.e. $f^{-1}(B_0^c) = f^{-1}(B_0)^c$.

(c) Prove that f respects unions, i.e. $f(A_0 \cup A_1) = f(A_0) \cup f(A_1)$.

(d) Show that $f(A_0^c) \subseteq f(A_0)^c$ and $f(A_0^c) \supseteq f(A_0)^c$ don't hold in general by constructing a counterexample to each.